GRAPH-BASED MODELS FOR WOODWINDS

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ABSTRACT

A model for woodwinds with tone and register holes is presented. It is inspired by the original idea of A.H. Benade considering the set of toneholes as a sequence of 'T-shaped' sections. This idea can be deepened thanks to more recent works on mathematical modelling and analysis of repetitive structures such as networks of strings, beams, membranes, pipes or canals. An essential feature of the model is that it keeps at the one dimensional level. The purpose of this work is to built upon the idea of Benade, inside a precise mathematical framework using concepts and methods from graph theory, for modelling the bore of woodwinds together with its holes in order to address questions like length corrections due to the lattice of (closed or open) toneholes or toneholes interactions. The long term objective is to use this type of model for simulation, characterization of the natural frequencies of woodwinds and, in a control theoretical setting, for musical acoustics questions such as design as well as for theoretical purposes. Applications of the approach are exposed as a program for future research.

1. INTRODUCTION

Wind instruments have been the subject of numerous studies for a long time and are now rather well understood [1–4]. A woodwind can be considered in the first approximation as a duct, the length of which can be made variable by opening or closing one or several toneholes, thereby adjusting the pitch of the instrument. But the acoustical behavior of woodwinds is known to be strongly influenced by the design of its system of tone holes, their sizes and spacings, while these last two parameters cannot be designed independently for a tube of a given thickness [5]. One first straightforward way of studying and simulating these instruments is to write the linear wave equation with convenient initial, boundary conditions on a domain that nevertheless can be somewhat complex. Nevertheless, this direct approach may not give sufficient enlightening to phenomena that interest the musician, the instrument maker or the acoustical physicist and especially how to design a woodwind for musical purposes. Therefore, A.H. Benade proposed to look at a woodwind as a sequence of what he called 'T-shaped sections', each consisting of a piece of the main bore and one tonehole, with given radii and lengths. In order to obtain qualitative results, a simplified version was used with identical such T-shaped sections, as a kind of periodic medium. Several effects were studied such as those due to the closed-holes or open-holes length corrections, to fork fingerings, or to the function of register hole for higher register functioning ([2] and [3], chap. 7 for a survey). The main objectives are to obtain information about the playing frequencies and the influence toneholes characteristics have on them. These characteristics essentially are of a geometric nature (size, spacing along the main duct). This approach has been the main basis of most works since (see the recent [6–8] e.g.). On another hand, the effects of discontinuities in acoustical ducts (either musical or not) have been investigated in [9] for the case of junctions of two, three and four guides, using modal decomposition and conformal maps. A second aspect in the study of wind instruments (brasses or woodwinds) concerns the bore cross-section. Although it is admitted that the only musically useful bores are members of the Bessel horn family (including cylindrical and conical bores), practical instruments show that the bore is not precisely cylindrical nor conical and that small variations from these idealized shapes arise, e.g. from deliberate alterations brought by the instrument maker in order to improve the tone or the tuning of an instrument [5]. This, together with the presence of toneholes or register holes, affects the natural frequencies of the whole instrument [6,8]. Thus it is interesting to have means to study this question and the related design problem in a precise way.

For all these questions, the commonly used way of studying woodwinds is through a modal approach, using the electric-acoustic analogy with equivalent circuits and their impedances within the transmission lines formalism, while approximating a duct as a sequence of cylinders or cones ; this is quite natural as musical instruments work usually in harmonic regimes. Mode matching is then used to make coherent the different modal decompositions.

In the present work, one takes for modelling a route different from the above-mentionned. Hopefully it will allow to give another light to the above questions and answer some other questions that still remain open. At the present stage, the results are of a theoretical nature but look promising. The original idea of A.H. Benade [5] was to consider a woodwind as a lattice or network of several tubes connected together with junctions. The present work is an attempt to build upon this idea in a mathematically precise way, with one main difference : it keeps a (one dimensional) PDE formalism instead of turning to the discrete, lumped-parameter, transmission lines framework. To this

end, the main ingredient comes from mathematical studies of equations on networks [10–13] and more recently on assemblages of several similar components such as strings, beams, plates, membranes (see the monograph [14] and references therein), pipes or canals [15] and, more specifically connected to the present work, vibrating systems. The basic idea is to consider the graph of this network connecting together the elementary components and to study for example the spectrum of this set through properties of the graph itself and of the components. Hence, the *skeleton* of a given woodwind can remain the same, whereas the model for individual components can be changed according to what effects (e.g. linear vs nonlinear) are to be studied. This way, one remains in the one dimensional setting although complex geometries are in order. Together with this modelling approach, we adopt a control theoretical point of view, considering the geometric parameters such as the duct and toneholes cross-sections or spacings between toneholes, that are indeed design parameters, as control parameters that have to be optimized in some sense that will be detailed in future work. A first step in that direction was done in [16] where the duct cross-section of general wind instruments without toneholes was considered as a control parameter for musical design purposes. This allowed to look at bore design as an optimal control problem. In a similar fashion, the cross-section of tone holes and their spacing can be considered as control parameters for design. We restrict in this work to linear acoustics, although the approach can be extended to a nonlinear context.
The presentation is organized as follows : in a first step, we recall the linear model for a unique duct, that can be of non uniform cross-section. In a second step (section 3), a model of a woodwind as a simple network of elementary components is presented. Then, in section 4, we show how to compute the natural frequencies in that context. Last, applications and future works such as design problems are presented and briefly discussed within a control theoretical setting.

2. LINEAR MODEL IN ONE DUCT

For the sake of self-containedness, we recall here the well-known one dimensional linear model (without sources), under different appearances, that is used for studying propagation in ducts, while making appear what we consider here as *control parameters*. The fluid is assumed to be barotropic i.e. the pressure is a function of the density ρ only : $p = p(\rho)$ and in the usual conditions of linear acoustics dealing with small perturbations of variables about their mean values. In the sequel, ρ_0 is the density of the gas at rest, c the velocity of sound, $p(x,t)$ the pressure, $v(x,t)$ the particular velocity, $A(x)$ the cross-section area of a tube at abcissa x. The fluid is as usual assumed irrotational, i.e. there exists a velocity potential ϕ : $v = \partial_x \phi$. A model for the plane wave propagation inside a one dimensional non uniform acoustic wave guide is the following [3] :

$$\left\{ \begin{array}{rcl} \partial_t v + \frac{A}{\rho_0}\partial_x p & = & 0 \\ \partial_t p + \frac{\rho_0 c^2}{A}\partial_x v & = & 0 \end{array} \right. \tag{1}$$

Losses can be taken into account by introducing an inhomogeneous part in the second member of this system. This PDE system can be reduced through differential elimination to the well-known horn equation for the pressure :

$$\frac{1}{c^2}\partial_{tt} p - \frac{1}{A}\partial_x (A \partial_x p) \quad = \quad 0 \tag{2}$$

which is also valid for the velocity potential. When the cross-section is constant, i.e. for cylindrical ducts, it reduces to the wave equation : $\frac{1}{c^2}\partial_{tt} p - \partial_{xx} p = 0$. Considering the cross-section area A as a design parameter or, in a control theoretical setting as a *control parameter*, equations (1) or (2) constitute a *hyperbolic partial differential control system*.

3. GRAPH BASED MODELS OF WOODWINDS

For modelling woodwinds within graph theory, consider the *skeleton* of a woodwind as in figure 1 : a scheme of a main tube (the duct of the woodwind) with several other tubes (the toneholes or the register hole) joining it at different locations is constituted of edges of a graph that meet at vertices (or nodes). One elementary situation with one

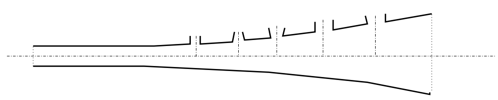

Figure 1. scheme of a woodwind with its graph

main duct and one tonehole appears in figure 2. As one can

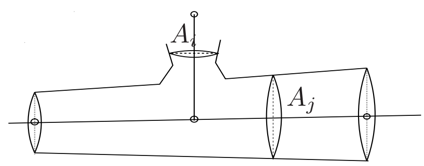

Figure 2. scheme of a woodwind with one tonehole

see on figure 2, each section is not 'T-shaped' in the terminology of A.H. Benade : in such a section, two tubes with different, non uniform cross-sections join together. This is in contrast with usual models that are cylindrical. One hopes to have this way a finer description of what happens e.g. when 'undercutting' is done for toneholes, which amounts to having non cylindrical toneholes.

3.1 Graph description

For the graph description, one follows closely [12] (see [17] e.g. for graph theory). We consider (see figure 3) that each portion of the main duct between two adjacent toneholes is modelled in a schematic way by an edge with two ends modelled by two vertices. Each tonehole or register hole is modelled schematically as an edge joining two vertices. This is relevant for one dimensional models considered here. Then the union of all these edges and vertices constitutes a graph. Observe first that the graph so associated to a woodwind is of a very special type : it is a *tree* as it is connected and contains no cycle [17]. The resulting underlying tree of a general woodwind is illustrated in figure 3 : for a wind instrument with n holes (tone and register holes), the associated tree has $N = 2n + 2$ vertices (or nodes), denoted V_i, and $N - 1 = 2n + 1$ edges, denoted E_i. Each edge and its associated quanti-

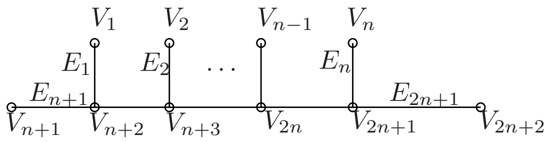

Figure 3. Graph of a woodwind with toneholes

ties are indexed by an integer : $i \in \mathcal{I} = \{1, \dots, N-1\}$. Therefore, one defines for each edge, the length l_i, the running variable $x_i \in [0, l_i]$, the cross-section area A_i, the pressure p_i, the particle velocity v_i and the velocity potential ϕ_i ($v_i = \partial_{x_i}\phi_i$), $i \in \mathcal{I}$. One can also assume, for the sake of generality, that the sound velocity, c_i, is different in each tube, although this is likely not to be the general case in woodwinds. The locations of end points of each tube, i.e. the vertices of the tree, are labelled by $j \in \mathcal{J} = \{1, \dots N\}$. Looking at figure 3, one sees that simple vertices belong to the boundary, $\partial\mathbf{G}$ of the graph and that multiple vertices belong to its interior $\mathring{\mathbf{G}}$ ($\mathbf{G} = \mathring{\mathbf{G}}$ $\cup\partial\mathbf{G}$). Hence, one distinguishes multiple vertices, indexed by $j \in \mathcal{J}_M = \{n+2, n+3, \dots, 2n, 2n+1\}$, where several tubes meet, from simple vertices, indexed by $j \in \mathcal{J}_S = \{1, 2, \dots, n, n+1, 2n+2\}$, which are the external ends of the tubes. Notice that for all woodwinds, exactly three edges obviously meet at one multiple vertex. An *ocarina* could be modelled as several tubes that join at one vertex, the cavity, thus as one multiple vertex with more than three joining edges but in such an instrument, the flow cannot be considered one dimensional. For $j \in \mathcal{J}$, it is useful to define :

$$\mathcal{I}_j = \{i \in \mathcal{I} : \text{the } i\text{th tube meets the jth vertex}\}$$

For $i \in \mathcal{I}_j$, one sets $x_{ij} = 0$ or l_i corresponding to which end meets the other tubes at the jth vertex. One also sets $\epsilon_{ij} = 1$ if $x_{ij} = l_i$ or $\epsilon_{ij} = -1$ if $x_{ij} = 0$, useful for the purpose of integration by parts below and description of the set of natural frequencies in section 4.

3.2 A woodwind linear model

The horn equation (2) in a simple duct with non uniform cross-section can be derived using variational calculus (see [16], appendix a). For a woodwind with its toneholes and register holes, we follow the same approach to derive the system of equations that will model the dynamics inside the instrument. First, at internal nodes, one has continuity conditions :

$$\text{the } \phi_i(x_{ij}, t) \text{ are equal } \forall i \in \mathcal{I}_j, \forall j \in \mathcal{J}_M$$

and geometric conditions on the cross-section areas at the junctions, e.g. that they are equal for the sections of the main duct at internal nodes, when it has no discontinuities, which is the most frequent situation (see nevertheless [18]). For a general pressure field, the lagrangian action density at each time instant inside one duct is the difference between a kinetic term and a potential term. The one dimensional propagation hypothesis implies that each duct can be considered as a continuous stack of cross-sections $A_i(x_i)$, parameterized by the abcissa x_i. For each section $A_i(x_i)$ located at x_i along the i^{th} horn axis, a mean action density is computed as the integral of densities of the particles over the section. This leads to an expression proportional to the cross-section area, i.e. to $A_i(x_i)$. Firstly, the kinetic term in each duct writes :

$$T_i(x_i, t) = \int_{A_i} \frac{1}{2}\rho_0 v_i^2 d\sigma = A_i \frac{\rho_0}{2} v_i^2 = A_i \frac{\rho_0}{2} |\partial_{x_i}\phi_i|^2$$

Similarly, the potential energy term is given as :

$$U_i(x_i, t) = \int_{A_i} \frac{p_i^2}{2\rho_0 c_i^2} d\sigma = A_i \frac{p_i^2}{2\rho_0 c^2} = A_i \frac{\rho_0}{2} |\frac{\partial_t \phi_i}{c_i}|^2$$

Summing up over all the edges of the graph, the lagrangian action inside the complete instrument writes :

$$\mathcal{L}(\phi) = \sum_{i\in\mathcal{I}} \int_{t_0}^{t_1} \int_0^{l_i} L_i(x_i, t) dx dt$$

$$= \frac{\rho_0}{2} \sum_{i\in\mathcal{I}} \int_{t_0}^{t_1} \int_0^{l_i} A_i \left(|\partial_{x_i}\phi_i|^2 - |\frac{\partial_t \phi_i}{c_i}|^2 \right) dx dt$$

Varying this action while integrating by parts with respect to both t and x, assuming that variations $\delta\phi_i$ vanish for $t = t_0$ and $t = t_1$, one gets the variation of $\mathcal{L}(\phi)$:

$$\begin{aligned} &\tfrac{d}{d\epsilon}\mathcal{L}(\phi + \epsilon\delta\phi)_{|\epsilon=0} = \tfrac{\rho_0}{2} \textstyle\int_{t_0}^{t_1} \sum_{i\in\mathcal{I}} \left[A_i \partial_{x_i}\phi_i \delta\phi_i\right]_0^{l_i} \\ &+ \left[\textstyle\int_0^{l_i} \left(\tfrac{1}{c_i^2} A_i \partial_{tt}\phi_i - \partial_{x_i}(A_i \partial_{x_i}\phi_i) \right) \delta\phi_i dx_i \right] dt \end{aligned} \quad (3)$$

Applying Hamilton's stationary action principle, one first considers, for each $i \in \mathcal{I}$, variations $\delta\phi_i$ that vanish near the vertices while setting the other variations to zero. This leads to the dynamics inside each tube :

$$\tfrac{1}{c_i^2} A_i \partial_{tt}\phi_i - \partial_{x_i}(A_i \partial_{x_i}\phi_i) = 0 \ , i \in \mathcal{I} \quad (4)$$

i.e. the horn wave equation (2) in each tube. Second, one considers variations $\delta\phi_i$ with support concentrated about one particular vertex E_j , $j \in \mathcal{J}$. One gets :

$$\sum_{i\in\mathcal{I}_j} \epsilon_{ij} A_i(x_{ij}, t) \partial_{x_i}\phi_i(x_{ij}, t) = 0 \ , \forall j \in \mathcal{J} \quad (5)$$

For multiple nodes, one must only have $\delta\phi_k = \delta\phi_l$ ($\forall k, l \in \mathcal{I}_j$ with $j \in \mathcal{J}_M$). Condition (5) is then a Kirchoff-type condition, the meaning of which is the conservation of flow at each multiple node, which is a natural node condition. When considering the case of external simple nodes (i.e. there is only one $i \in \mathcal{I}_j$ with $j \in \mathcal{J}_S$) and with the reasonable assumption that $A_i(x_{ij}) \neq 0$, this gives the natural Neuman-type boundary conditions :

$$\epsilon_{ij} \partial_{x_i}\phi_i(x_{ij}, t) = 0 \quad (6)$$

corresponding to a closed hole. Imposed boundary conditions at the external simple nodes, corresponding to open holes radiation conditions, or to the excitation mechanism, can be added through the introduction of a suitable work functional into $\mathcal{L}(\phi)$. All the above represents a wind instrument without active components and for which losses can be taken into account in the boundary conditions. In the following section, using this model, we concentrate on computing the natural frequencies of a woodwind through a generalized eigenvalue problem.

4. NATURAL FREQUENCIES OF A WOODWIND

Determining the natural frequencies of a wind instrument - i.e., in mathematical terms, the spectrum of the differential operator defined in (4)- is central from the viewpoints of physics, of instrument making and is important for musical practice. One straightforward way for their computation is through a finite element approximation of the continuous underlying system, with possibly complicated boundaries, followed by solving the associated generalized eigenvalue problem with a convenient numerical algorithm. But this may be tricky and not much informative about the influence of geometric parameters such as hole sizes and inter-hole spacings. Thus approximating the natural frequencies by corrections from those of idealized bore shapes such as cylinders and cones has been for a long time the favoured approach in the musical acoustics community and several formulae have been given for that purpose [19, 20] . The main reason for this is that no real instrument has an exact cylindrical or conical bore shape [19] whereas for simple duct shapes, exact formulae are known. Thus it is worthwhile to investigate the normal modes of tubes which depart from these exact shapes [19, 21]. The effects of holes also greatly affects the natural frequencies of an instrument [6, 8]. In that context, the graph modelling approach presented above can be an interesting method for computing the natural frequencies, as it consists in one dimensional equations thus of quite lower complexity than full 3D models, while connecting it to geometric parameters of interest to instrument makers, such as tone hole dimensions and inter hole spacings.

In [12], it is shown that the structure of the underlying graph of an elliptic operator on a network plays a distinctive role in the spectrum of the associated eigenvalue problem. This is done using an equivalent boundary value problem for a matrix differential equation. This idea is at the basis of the present section. One essential aspect of the method is to rescale the spatial variables associated with each element of the structure to the uniform interval $[0, 1]$, while accounting for an orientation on each element. Then, a special matrix calculus - on an element by element basis- due to J. Hadamard, allows to pose and solve the corresponding eigenvalue problem leading to the searched spectrum.

First, using equations (4), (5) in section 3, the set of natural frequencies of a woodwind model is the solution of the following eigenvalue problem :

$$\begin{cases} \phi_i \in C^2([0,l_i]) \ , \forall i \in \mathcal{I} \ , \phi = (\phi_i) \\ \phi \text{ is continuous on } \mathbf{G} \\ \partial_{x_i x_i}\phi_i + \frac{\partial_{x_i} A_i}{A_i}\partial_{x_i}\phi_i = -\frac{\lambda^2}{c_i^2}\phi_i \ , \forall i \in \mathcal{I} \\ \sum_{i\in\mathcal{I}_j} \epsilon_{ij} A(x_{ij},t)\partial_{x_i}\phi_i(x_{ij},t) = 0 \ , \forall j \in \mathcal{J} \end{cases} \tag{7}$$

It can be formulated in a synthetic fashion as follows. Consider the tree (see figure 3), $\mathbf{G} \subset \mathbb{R}^2$, of a woodwind, with its set of N vertices, $V(\mathbf{G}) := \{V_i \ , i = 1, \ldots, N\}$, and its set of $N-1$ edges, $E(\mathbf{G}) := \{E_j \ , j = 1, \ldots, N-1\}$. The edges are parameterized by $\pi_j : [0, l_j] \to \mathbb{R}^2$, where the running variable $x_j \in [0, l_j]$ represents the arc length. The maps π_j are assumed to be C^2-smooth. We introduce the *incidence matrix* $\mathcal{D} = (d_{ij})$:

$$d_{ij} = \begin{cases} 1 & \text{if } \pi_j(l_j) = V_i \\ -1 & \text{if } \pi_j(0) = V_i \\ 0 & \text{elsewhere} \end{cases} \tag{8}$$

which is the matrix version of the ϵ_{ij}'s of the previous section, and the *adjacency matrix* $\mathcal{E} = (e_{ij})$:

$$e_{ih} = \begin{cases} 1 & \text{if there exists } s = s(i,h) \in \mathcal{I}, \\ & \text{with } k_s \cap V(\mathbf{G}) = \{V_i, V_h\}, \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

which describe how edges connect vertices. Whenever $e_{ih} = 0$, set $s(i,h) = 1$. For example, in case of one duct with one tonehole, these matrices are :

$$\mathcal{D} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{10}$$

for the orientation of the three edges for which the origin is at simple vertices, according to the scheme in figure 3, and :

$$\mathcal{E} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{11}$$

Now we recall the Hadamard operations for $n \times n$ matrices $P = (p_{ij})$: the product $P.Q$ is done element by element, $(P.Q)_{ij} = p_{ij} q_{ij}$ and for any function $f : \mathbb{R} \to \mathbb{R}$, the matrix $f(P)$ is given by $f(P) = (q_{ih})$ with :

$$q_{ih} = \begin{cases} f(p_{ih}) & \text{if } e_{ih} = 1 \\ 0 & \text{if } e_{ih} = 0 \end{cases} \tag{12}$$

and especially when $f(x) = x^r, r \in \mathbb{R}$ for the matrix powers P^r. Define the vector $e = (1, \ldots, 1)^T$ and, for any n-vector, $y = (y_i)$ the diagonal matrix diag(y)= $(\delta_{ij} y_i)$ with δ the Kronecker delta function. Define also the matrices :

$$\begin{aligned} \mathcal{A} = (A_{ih}) &= (A_{s(i,h)} e_{ih}) \\ \mathcal{C} = (c_{ih}) &= (c_{s(i,h)} e_{ih}) \\ \mathcal{L} = (l_{ih}) &= (l_{s(i,h)} e_{ih}) \end{aligned} \tag{13}$$

and for $\phi : \mathbf{G} \to \mathbb{R}$ and $x \in [0, 1]$, $\Phi(x) = (\phi_{ih}(x))$ with :

$$\phi_{ih}(x) = e_{ih} \phi_{s(i,h)} \left[l_{ih} \left(\frac{1 + d_{s(i,h)}}{2} - x d_{is(i,h)} \right) \right]$$

such that

$$\Phi(0) = \left(\phi_{ih}\left(\pi^{-1}_{s(i,h)}(E_i)\right)\right) = (\phi_i(x_{ij},t))\, e^T.\mathcal{E} = \psi e^T.\mathcal{E}$$

$\psi = (\phi_i(x_{ij},t))$ denoting the vector of values of ϕ_i's at the vertices. Notice the symmetry $\phi_{hi}(x) = \phi_{ih}(1-x), x \in [0,1]$. Last, as the independent variables x_i have all been rescaled to $[0,1]$, we denote the spatial derivatives with primes ($u^{'} = \partial_x u$) in the rest of this section, to conform with the usual notation. With this set of notations, the eigenvalue problem (7) is equivalent to the following :

$$\left\{\begin{array}{l} \phi_{ih} \in C^2([0,1]) \text{ and } (e_{ih} = 0 \Rightarrow \phi_{ih} = 0) \forall i,h \in \mathcal{I} \\ \mathcal{L}^{-2}.\mathcal{C}^2.\Phi^{''}(x) + \mathcal{L}^{-1}.\mathcal{C}^2.\mathcal{A}^{-1}.\mathcal{A}^{'}.\Phi^{'}(x) = -\lambda^2\Phi(x), \\ \forall x \in [0,1] \\ \exists \psi \in \mathbb{R}^N \ : \ \Phi(0) = \psi e^T.\mathcal{E} \\ \left[\mathcal{L}^{-1}.\mathcal{A}.\mathcal{C}.\Phi^{'}(0)\right] e = 0 \\ \Phi^T(x) = \Phi(1-x),\ \forall x \in [0,1] \end{array}\right. \tag{14}$$

The solutions of this problem furnish the natural frequencies of the modelled woodwind. A detailed analysis within this generality is deferred to future work. Instead, let us look here at an illustrative and important particular case, when all the c_i's are equal to the same constant c, which is the most frequent assumption. Assume also that all ducts are cylindrical, which implies : $\mathcal{A}^{'} = 0$. The corresponding eigenvalue problem reduces to :

$$\left\{\begin{array}{l} \phi_{ih} \in C^2([0,1]) \text{ and } (e_{ih} = 0 \Rightarrow \phi_{ih} = 0) \forall i,h \in \mathcal{I} \\ \mathcal{L}^{-2}.\Phi^{''}(x) = -\frac{\lambda^2}{c^2}\Phi(x)\ , \forall x \in [0,1] \\ \exists \psi \in \mathbb{R}^N \ : \ \Phi(0) = \psi e^T.\mathcal{E} \\ \left[\mathcal{L}^{-1}.\mathcal{A}.\Phi^{'}(0)\right] e = 0 \\ \Phi^T(x) = \Phi(1-x),\ \forall x \in [0,1] \end{array}\right. \tag{15}$$

Using the Hadamard calculus above, the solution of this problem can be given explicitely as :

$$\Phi(x) = \cos(\frac{x\lambda}{c}\mathcal{L}).\Phi(0) + \frac{c}{\lambda}\mathcal{L}^{-1}.\sin(\frac{x\lambda}{c}\mathcal{L}).\Phi^{'}(0)$$

Thus one has an explicit expression of the eigenvector $\Phi(x)$ corresponding to an eigenvalue λ. This is very important as it gives the solution of a somewhat complex eigenvalue problem in a comprehensible form that moreover can be related to existing results for simpler systems as a simple cylinder, for the sake of comparisons for example. From this, the detailed structure of the set of natural frequencies can be 'read into' the structure of the underlying tree, through the structure of the matrices. Omitting the demonstrations, set $\mathcal{B} = \frac{1}{c}\mathcal{L}$ and define the matrix :

$$\mathcal{M}(\lambda) = \mathcal{A}(\sin(\lambda\mathcal{B}))^{-1} - \text{diag}\left[\mathcal{A}(\sin(\lambda\mathcal{B}))^{-1}\cos(\lambda\mathcal{B})e\right]$$

with given physical and geometric parameters $c, \mathcal{L}, \mathcal{A}, \mathcal{E}$. Thanks to the expression of $\Phi(x)$ and to the boundary conditions in(15), the eigenvalues of problem (15) can be shown to be of one of the following two types :

1. $\lambda = l^{-1}_{s(i,h)} c\pi k$ for some $i,h = 1\ldots,N$; $k \in \mathbb{Z}$ which are the eigenvalues of elementary ducts.
2. λ is a solution of the transcental equation :

$$\det \mathcal{M}(\lambda) = 0 \tag{16}$$

Therefore the complete set of natural frequencies of the windwood model is explicited. These results parallel those for a parabolic problem on networks in [12], extended in [14] for networks of hyperbolic mechanical systems made of strings and beams.

5. DISCUSSION

The model developped in section 3 accounts in a simple way for important geometric parameters : toneholes spacings are given by the length of internal edges $E_k, k = n+2,\ldots,2n+1$ and height of toneholes are given by the length of external edges $E_k, k = 1,\ldots,n$ (see figure 3), all gathered in the matrix $\mathcal{L}$. The diameter of these last ones is explicitely given in the dynamic equations ((4), (5), (7) or (14). The main duct diameter is given by the sequence of diameters corresponding to the internal edges. All diameters are elements of matrix $\mathcal{A}$. Thus the set of natural frequencies of a woodwind (the partials of the duct) can be computed and varied as a function of all these parameters. One point that is not clear at the moment is how to describe the geometry at the junctions. This model makes it possible to study anew usual questions in musical acoustics related to the natural frequencies of woodwinds. One can mention as first examples : quantify length corrections due to the closed holes or open holes lattices ; study the case of one main duct with one tonehole, as in [6] ; analyze the effect of different cross-fingerings on the playing frequency ; quantify the experimental observation that the tuning properties of a woodwind are predominantly affected by the properties of only the first two or three open toneholes [5]. Due to the one dimensional nature of the model, the complexity of computing the natural frequencies or of a simulation with the presented model is low : for an instrument with 8 toneholes, the corresponding model is made of 18 equations for the dynamics and as much unknowns. These points and comparisons with the usual transmission lines approach will be investigated in future works.

5.1 Woodwinds design as a control problem

As it has been shown in [16], focussing on bore shape design, control theory can be a useful framework for design problems in musical acoustics. The model that has been presented in section 3 allows to pursue this line of investigation by including in the design process important geometric parameters such as toneholes dimensions and spacings between them. These parameters, i.e. the matrices $\mathcal{A}$ and $\mathcal{L}$, can be considered as control parameters for a series of inverse problems. Optimal control theory can then be used as in [16] when a suitable optimization criterion is defined. A typical one for woodwinds is the precise alignement of fundamental frequencies for the first and second registers. One important point is that such design problems make appear controls that are *distributed* in the space variable. On the contrary, previous works on network models of pipes or canals, based on the nonlinear St Venant equations, focussed on boundary control in the time variable [15, 22], because the geometry of canals was there given and fixed. The above model for woodwinds can be also the subject of initial-boundary control problems for the purpose of simulation. Thus several questions about woodwinds lead with this formulation either to boundary or distributed control problems.

6. CONCLUSION AND FUTURE RESEARCH

Modelling a woodwind using graph-theoretical concepts opens new possibilities to treat questions such as length corrections and can be useful for simulation. The computational complexity of the model is relatively low. Nevertheless it is likely that it cannot compete with the transmission lines approach for real-time sound synthesis. But we think that for off-line analysis and for design it can be helpful. Also, it allows to treat design questions as control problems hence can be useful as a tool for instrument making as well as for better insight into the physics of the instruments. Future research will focus on the relationship between the above effects and the geometric parameters, in the same spirit as in [5] and recent research [8]. In that respect, the matrix formulation of the present work fits well a perturbation analysis, useful for studying the influence of geometry on the natural frequencies through the matrix $\mathcal{M}$. Also, the excitation mechanism and related questions have to and will be accounted for in this model.

The previous developments have been limited to linear models of elementary ducts because the natural frequencies is an utmost important characteristic of a wind instrument. But it is known that several nonlinear effects appear too in playing situations. The graph-based approach can be extended straightforwardly to the nonlinear situation, at the price of a greater complexity, by considering the nonlinear equations in an elementary duct, together with the same tree *skeleton*. This is currently under investigation.

7. REFERENCES

[1] N. Fletcher and T. Rossing, *The physics of musical instruments*. Springer, 1991.

[2] A. H. Benade, *Fundamentals of musical acoustics*. Dover, 1990, 2d edition.

[3] A. Chaigne and J. Kergomard, *Acoustique des instruments de musique*. Belin, 2008.

[4] H. Bouasse, *Instruments à vent, I, II*. Delagrave, 1929, new edition Librairie A. Blanchard, 1986.

[5] A. Benade, “On the mathematical theory of woodwind finger hole,” *J. Acoust. Soc. Amer.*, vol. 32, pp. 1591–1608, 1960.

[6] V. Dubos, J. Kergomard, A. Khettabi, J. P. Dalmont, D. H. Keefe, and C. J. Nederveen, “Theory of sound propagation in a duct with a branched tube using modal decomposition,” *Acta Acustica*, vol. 85, pp. 153–169, 1999.

[7] E. Moers and J. Kergomard, “On the cutoff frequency of clarinet-like instruments. geometrical versus acoustical regularity,” *Acta Acustica u. Acustica*, vol. 97, no. 6, pp. 984–996, 2011.

[8] A. Lefebvre, G. P. Scavone, and J. Kergomard, “External toneholes interaction in woodwind instruments,” 2012, preprint, http://arXiv.org/pdf/1207.5490 or http://hal.archives-ouvertes.fr/hal-00720132.

[9] J. Kergomard, A. Khettabi, and A. Garcia, “Formulation générale des jonctions de guides d’ondes aux basses fréquences,” *C.R. Acad. Sci. Paris*, vol. 319, no. II, pp. 887–892, 1994.

[10] W. H. Wittrick and F. W. Williams, “Natural frequencies of repetitive structures,” *Quart. J. Mech. and Applied Math.*, vol. XXIV, no. 3, pp. 285–308, 1971.

[11] S. Nicaise, “Some results on spectral theory over networks applied to nerve impulse transmission,” in *Lecture notes n mathematics*, C. Brezinski, Ed., 1985, pp. 533–541.

[12] J. Von Below, “A characteristic equation associated to an eigenvalue problem on c^2 networks,” *Linear algebra and its applications*, vol. 71, pp. 309–325, 1985.

[13] ——, “Classical solvability of linear equations on networks,” *J. Diff. Eqns*, vol. 72, pp. 316–337, 1988.

[14] J. L. Lagnese, G. Leugering, and E. J. P. G. Schmidt, *Modeling, analysis and control of dynamic elastic multilink structures*. Birkhaüser, 1994.

[15] G. Leugering and E. J. P. G. Schmidt, “On the modelling and stabilization of flows in networks of open canals,” *SIAM J. Control Optim.*, vol. 41, no. 1, pp. 164–180, 2002.

[16] G. Le Vey, “Optimal control theory : a method for the design of wind instruments,” *Acta Acustica u. Acustica*, vol. 96, pp. 722–732, 2010.

[17] J. W. Wilson, *Introduction to graph theory*. Academic Press, 1972.

[18] J. P. Dalmont and J. Kergomard, “Lattices of sound tubes with harmonically related eigenfrequencies,” *Acta Acustica*, vol. 2, no. 5, pp. 421–430, 1994.

[19] A. H. Benade, “On woodwind instruments bores,” *J. Acoust. Soc. Amer.*, vol. 31, no. 2, pp. 137–146, 1959.

[20] A. H. Benade and E. V. Jansson, “On plane and spherical waves in horns with non uniform flare,” *Acustica*, vol. 31, no. 2, pp. 79–98, 1974.

[21] H. Bouasse, *Tuyaux et résonateurs*. Delagrave, 1929, new edition Librairie A. Blanchard, 1986.

[22] J. de Halleux, C. Prieur, J. M. Coron, B. d’Andréa Novel, and G. Bastin, “Boundary feedback control in networks of open channels,” *Automatica*, vol. 39, pp. 1365–1376, 2003.